Energy in Dilaton Gravity in Canonical Approach

M.Z.Iofa
Nuclear Physics Institute
Moscow State University
Moscow 119899, Russia

Abstract

An expression for energy in string-inspired dilaton gravity is obtained in canonical approach.

The problem of definition of energy in general relativity has a long history and was repeatedly discussed in different aspects [1-6]. In standard gravity interacting with matter, it was shown that for field configurations with asymptotically flat metric with components sufficiently rapidly decreasing at spacial infinity one can define a nonnegative expression which is naturally interpreted as energy. In canonical approach, energy is defined as the value of the standard hamiltonian taken on the shell of zero constraints [2, 4, 5].

$$E = H|_{\{\Phi\}=0} \tag{1}$$

and has a functional form of a total spacial divergence

$$E = -\int \partial_i q^i d^D x$$

(greek and latin indices take $D+1$ and D values respectively).

In this note, following the methods of canonical approach to standard gravity [2, 4, 5], we discuss an expression for energy in dilaton gravity which appears as effective field theory for closed bosonic string theory. This expression can be used for calculation of mass of recently studied exact solutions in dilaton gravity such as black strings etc.

In $O(\alpha')$ order, for genus zero surfaces, effective action of closed bosonic string theory is [7]

$$S' = -\int d^{D+1}x e^{-\Phi}\sqrt{|g|}\left(R - (D\Phi)^2 + 2D^2\Phi + \frac{H^2}{12} + \Lambda\right) \tag{2}$$

(in the following, we do not write the term $\frac{H^2}{12}$ irrelevant for our discussion explicitly).

Let the dynamical fields be $h_{\mu\nu}$ and φ, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\Phi = \Phi^0 + \varphi$. Here $(\eta_{\mu\nu}, \Phi^0)$ is the vacuum (flat space) solution. Following the standard approach, we rewrite (2) as the sum of two terms. The first term contains only first-order derivatives, the second one is the total derivative and produces terms with second-order derivatives. Using the relation $\sqrt{|g|}R = \sqrt{|g|}G + \partial_\lambda(\sqrt{|g|}w^\lambda)$ [3], (2) is rewritten as

$$\begin{aligned} S' \quad = \quad & -\int d^{D+1}x e^{-\Phi}\sqrt{|g|}\left(G + \Lambda - (D\Phi)^2 + w^\lambda\partial_\lambda\Phi + 2D^2\Phi^0 + 2D_\lambda\Phi D^\lambda\varphi\right) - \\ & -\int d^{D+1}x \partial_\lambda\left(\sqrt{|g|}e^{-\Phi}(w^\lambda + 2D^\lambda\varphi)\right). \end{aligned} \tag{3}$$

Here

$$w^\lambda = g^{\mu\nu}\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\Gamma^\nu_{\mu\nu} = g_{\mu\rho,\nu}G^{\mu\nu\lambda\rho} \tag{4}$$

where

$$G^{\mu\nu\lambda\rho} = -g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\lambda\nu}$$

Omitting the term with the total derivative, we obtain the action with only first-order derivatives of dynamical fields:

$$S = -\int d^{D+1}x e^{-\Phi}\sqrt{|g|}\left(G + \Lambda - (D\Phi)^2 + w^\lambda\partial_\lambda\Phi + 2D^2\Phi^0 + 2D_\lambda\Phi D^\lambda\varphi\right). \tag{5}$$

The fields $(g_{\mu\nu}, \Phi)$ are assumed to be asymptotic to the static vacuum solution $(\eta_{\mu\nu}, \Phi^0)$, and dynamical fields are supposed to decrease at spacial infinity, so that the following transformations and formulas make

sence. To make formulas more transparent (having in view, as an example, applications to black string solutions of the gauged $SL(2,R)\times U(1)^N/U(1)$ models [8, 9, 10]), we consider the case of configurations with $g_{0i}=0$.

Let us construct the hamiltonian for the lagrangian density (5). The components of w^λ are

$$\begin{aligned} w^0 &= \dot h_{ls}G^{ls00}+\bar w^0 \\ w^m &= \dot h_{0l}G^{m0l0}+\dot h_{ls}G^{lsm0}+\bar w^m \end{aligned} \tag{6}$$

Here the dots stand for x^0-derivatives, $\bar w^\lambda$ contain no x^0-derivatives. Velocities-dependent part of the lagrangian density is

$$L_1=-e^{-\Phi}a\left[-\frac{1}{4}\dot h_{ik}G^{iklm}\dot h_{lm}-\dot h_{ls}g^{ls}\dot\varphi+\dot\varphi^2-g^{ik}\left(\partial_i\ln a-\partial_i\Phi\right)\dot h_{0k}\right]. \tag{7}$$

Here

$$a=\frac{\gamma}{\kappa},\qquad \gamma=\sqrt{|detg_{ik}|},\qquad \kappa=\frac{1}{\sqrt{g^{00}}}.$$

The signature of the metric is $(+--\cdots)$. The momenta are

$$\begin{aligned} &p^{00}=0, \\ &p^{0k}=-g^{ik}\partial_i(ae^{-\Phi}), \\ &p^{ik}=\tfrac{1}{2}ae^{-\Phi}\left[(\dot h_{lm}g^{lm}-2\dot\varphi)g^{ik}+\dot h^{ik}\right], \\ &p_\varphi=ae^{-\Phi}(2\dot\varphi-\dot h_{lm}g^{lm}). \end{aligned} \tag{8}$$

Velocities $\dot h_{0i}$ are not expressed through the momenta and yield the primary constraints

$$\begin{aligned} &\Phi^{(1)0}=p^{00}=0, \\ &\Phi^{(1)k}=p^{0k}+g^{ik}\partial_i(ae^{-\Phi})=0. \end{aligned} \tag{9}$$

Momenta-dependent part of the hamiltonian density is

$$H_1=e^{-\Phi}\frac{\kappa}{\gamma}\left[p^{ik}g_{il}g_{km}p^{lm}+p^{ik}g_{ik}p_\varphi+\frac{D-1}{4}p_\varphi^2\right]. \tag{10}$$

Velocities-independent part of the lagrangian can be written as

$$\begin{aligned} L_2 \;=\; & -e^{-\Phi}\kappa\gamma\left[R_D+\Lambda-D_i\Phi^0D^i\Phi^0+2D_iD^i\Phi^0+D_i\varphi D^i\varphi+(\bar w^l-\hat w^l)\partial_l\Phi\right]+ \\ & +\partial_l(e^{-\Phi}\kappa\gamma\hat w^l). \end{aligned} \tag{11}$$

Here $\bar w^l$ and $\hat w^l$ are

$$\begin{aligned} \bar w^l &= -\gamma\kappa\left[2g^{ln}\partial_n\ln\gamma\kappa+\partial_ng^{ln}\right] \\ \hat w^l &= -\gamma\left[2g^{ln}\partial_n\ln\gamma+\partial_ng^{ln}\right], \end{aligned} \tag{12}$$

$\sqrt{g}=\gamma\kappa$, R_D is the curvature scalar constructed from spacial components of the metric g_{ik}. Rearranging in (11) the terms containing derivatives of κ (which come from $D^iD_i\Phi^0$ and $(\bar w^l-\hat w^l)\partial_l\Phi$) so that κ enters either without derivatives or as a factor in the total divergence of some expression, we get

$$\begin{aligned} L_2 \;=\; & -e^{-\Phi}\sqrt{|g|}\left[R_D+\Lambda-\nabla_i\Phi\nabla^i\Phi+2\nabla_i\nabla^i\Phi\right]+ \\ & +\partial_l(e^{-\Phi}\sqrt{|g|}(\hat w^l+2g^{lk}\partial_k\varphi)). \end{aligned} \tag{13}$$

Here ∇_i is covariant derivative constructed from spacial components of the metric. Note that the first term in (13) coincides with the 00 component of the equation of motion derived from (2) or (3) taken at the surface of constant x^0.

Writing the second term in (13) as

$$e^{-\Phi}\gamma(\kappa-1)(\hat{w}^l+2\nabla^l\varphi)+e^{-\Phi}\gamma(\hat{w}^l+2\nabla^l\varphi) \tag{14}$$

and provided the combination $e^{-\Phi}\gamma(\kappa-1)(\hat{w}^l+2\nabla^l\varphi)$ sufficiently rapidly vanishes at spacial infinity (which is true for explicit examples of interest mentioned above), we obtain that the space integral of the total divergence from the first term in (14) vanishes and in (13) this term can be omitted.

Collecting (10) and (13), we obtain the hamiltonian density

$$H=\kappa T^0-\partial_l(e^{-\Phi}\gamma(\hat{w}^l+2\nabla^l\varphi)), \tag{15}$$

where

$$\begin{aligned} T^0 \quad = \quad & \kappa\left\{e^{-\Phi}\frac{1}{\gamma}\left[p^{ik}g_{il}g_{km}p^{lm}+p^{ik}g_{ik}p_\varphi+\frac{D-1}{4}p_\varphi^2\right]+\right. \\ & \left.+\,e^{-\Phi}\gamma\left[R_D+\Lambda-\nabla_i\Phi\nabla^i\Phi+2\nabla_i\nabla^i\Phi\right]\right\} \end{aligned} \tag{16}$$

and [5]

$$H^{(1)}=H+\lambda_\mu p^{0\mu}.$$

In general case when $g_{0i}\neq 0$, there appear terms $h_{0i}T^i$, and now T^0 and T^i depend on g_{0i}. As in standard gravity, the functions λ_μ cannot be determined from conditions of time conservation of the primary constraints

$$\left\{\Phi^{\mu(1)},H^{(1)}\right\}=0.$$

Instead, secondary constraints appear, which can be chosen as linear combination of constraints T^μ (cf. [4, 5]). The set of constraints is defined by the gauge (gravitational) part of the action, and by adding the dilaton one does not change the gauge structure of the action. This follows also from the fact that in terms of the new metric $\bar{g}_{\mu\nu}=g_{\mu\nu}e^{\frac{-2\Phi}{D-1}}$ the kinetic part of the action (2) assumes the standard form (however, in this case, the metric $\bar{g}_{\mu\nu}$ is asymptotically nonflat).

Defining the energy as in (1), we finally have

$$E=-\int d^Dx\partial_l(e^{-\Phi}\gamma(\hat{w}^l+2\nabla^l\varphi)). \tag{17}$$

In particular case of black-string solutions of gauged $SL(2,R)\times U(1)^N/U(1)$ WZW models [8-10], it can be verified that the fields decrease sufficiently rapidly in directions asymptotically transverse to the string to make all the above transformations valid. In this case, (17) should be interpreted as the energy of the string per unit length. For $N=1$ the formula (17) coincides with the expression used in [8].

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